

A NOTE ON THE TRANSCENDENCE OF INFINITE PRODUCTS

JAROSLAV HANČL,¹ ONDŘEJ KOLOUCH,² SIMONA PULCEROVÁ,
JAN ŠTĚPNIČKA,² Ostrava

(Received September 28, 2010)

Abstract. The paper deals with several criteria for the transcendence of infinite products of the form $\prod_{n=1}^{\infty} [b_n \alpha^{a_n}] / b_n \alpha^{a_n}$ where $\alpha > 1$ is a positive algebraic number having a conjugate α^* such that $\alpha \neq |\alpha^*| > 1$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers with some specific conditions.

The proofs are based on the recent theorem of Corvaja and Zannier which relies on the Subspace Theorem (P. Corvaja, U. Zannier: On the rational approximation to the powers of an algebraic number: solution of two problems of Mahler and Mendès France, Acta Math. 193, (2004), 175–191).

Keywords: transcendence, infinite product

MSC 2010: 11J81

1. INTRODUCTION

Following Erdős [4], Corvaja and Zannier [2] we prove

Theorem 1. *The number $x = \prod_{n=1}^{\infty} [n(\sqrt{5} + 1)^{k^n}] / n(\sqrt{5} + 1)^{k^n}$ is transcendental for all integers k greater than 4.*

Here $[z]$ means the integer part of the number z . The authors do not know if the number x is also transcendental or irrational for $k = 2, 3$ and 4.

¹ This paper has been elaborated in the framework of the IT4Innovations Centre of Excellence project, reg. no. CZ.1.05/1.1.00/02.0070 supported by Operational Programme ‘Research and Development for Innovations’ funded by Structural Funds of the European Union and state budget of the Czech Republic and by grants no. ME09017, P201/12/2351 and MSM 6198898701.

² The authors were supported by the grant 01798/2011/RRC of the Moravian-Silesian region.

In 2000 Zhu [14] proved some criteria for an infinite product to be transcendental. Making use of linear recurrence sequences of the second order Nyblom [11] constructed a set of transcendental valued infinite products. Utilizing theta series Kim [9] and Koo described some interesting infinite products. Recently Corvaja and Hančl [1] established a criterion for an infinite product to be transcendental. Tachiya [12] found some transcendental valued infinite products of algebraic numbers. Zhou [13] worked with similar products and obtained some irrationality results. All this shows that metric properties of infinite products are of current interest.

Erdős [4] proved that if $a = \{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $\liminf_{n \rightarrow \infty} a_n^{1/2^n} = \infty$ then the expressible set $E_a = \left\{ \sum_{n=1}^{\infty} 1/a_n c_n, c_n \in \mathbb{N} \right\}$ does not contain rational numbers. Using this idea of Erdős, Hančl, Nair and Šustek [6] found some necessary conditions for the Lebesgue measure of E_a to be equal to zero. For other applications of the method of Erdős see e.g. [5], [7] or [8]. It seems likely that this method still has great potential.

Our main theorem is Theorem 2. Its proof makes use of the main theorem in [2]. See also [3]. Theorem 2 and the method of Erdős yield Theorems 3–7. In all of Theorems 2–7 we suppose that α is a positive algebraic number greater than 1 having a conjugate α^* such that $\alpha \neq |\alpha^*| > 1$ where $|z|$ means the usual absolute value of the number z . Denote by \mathbb{N} and \mathbb{Q} the set of all natural and rational numbers, respectively. If α is an algebraic number then set $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, the degree of the algebraic number field $\mathbb{Q}(\alpha)$.

2. MAIN RESULTS

Theorem 2. *Let x and γ be real numbers such that $\gamma > 0$. If for infinitely many positive integers n, p and q*

$$(2.1) \quad 0 < \left| x - \frac{p}{q\alpha^n} \right| < \frac{1}{\alpha^{n(1+\gamma)} q^{1+\gamma+d}},$$

then the number x is transcendental.

Theorem 3. *Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > 2$. Then the number $x = \prod_{n=1}^{\infty} [\alpha^{a_n}] / \alpha^{a_n}$ is transcendental.*

Theorem 4. *Let $\varepsilon > 0$. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers with $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > 2 + 1/\varepsilon$. Assume that $\alpha^{a_n} > n^{1+\varepsilon}$ for every sufficiently large n . Then the number $x = \prod_{n=1}^{\infty} [\alpha^{a_n}] / \alpha^{a_n}$ is transcendental.*

Theorem 5. Let δ and ε be two positive real numbers. Assume that

$$(2.2) \quad \frac{1+d+\delta}{1+d} \cdot \frac{\varepsilon}{1+\varepsilon} > 1.$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that the sequence $\{B_n\}_{n=1}^{\infty} = \{b_n \alpha^{a_n}\}_{n=1}^{\infty}$ is non-decreasing and

$$(2.3) \quad \limsup_{n \rightarrow \infty} B_n^{1/(2+d+\delta)^n} = \infty.$$

Assume that $B_n > n^{1+\varepsilon}$ for every sufficiently large n . Then the number $x = \prod_{n=1}^{\infty} [B_n]/B_n = \prod_{n=1}^{\infty} [b_n \alpha^{a_n}]/b_n \alpha^{a_n}$ is transcendental.

Theorem 6. Assume that s is a non-negative real number. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is strictly increasing, $\{B_n\}_{n=1}^{\infty} = \{b_n \alpha^{a_n}\}_{n=1}^{\infty}$ is non-decreasing,

$$(2.4) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > 2 + \frac{sd}{s+1}$$

and

$$(2.5) \quad b_n = \alpha^{sa_n} + o(\alpha^{sa_n}).$$

Then the number $x = \prod_{n=1}^{\infty} [B_n]/B_n = \prod_{n=1}^{\infty} [b_n \alpha^{a_n}]/b_n \alpha^{a_n}$ is transcendental.

Theorem 7. Assume that ε and s are real numbers with $s \geq 0$ and $\varepsilon > 0$. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing,

$$(2.6) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > \left(1 + \frac{sd}{s+1}\right) \left(1 + \frac{1}{\varepsilon}\right) + 1,$$

$$(2.7) \quad \alpha^{a_n} > n^{1+\varepsilon}$$

and

$$(2.8) \quad b_n = \alpha^{sa_n} + o(\alpha^{sa_n}).$$

Then the number $x = \prod_{n=1}^{\infty} [b_n \alpha^{a_n}]/b_n \alpha^{a_n}$ is transcendental.

3. PROOFS

Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorem 5. It is enough to set $\alpha = \sqrt{5} + 1$, $\varepsilon = 7$, $\delta = \frac{1}{2}$, $b_n = n$ and $a_n = 5^n$ for all $n \in \mathbb{N}$. Then $d = 2$ and α has only one conjugate $\alpha^* = -\sqrt{5} + 1$. \square

Proof of Theorem 2. In fact Theorem 2 is a consequence of the main theorem in [2]. Assume Theorem 2 does not hold. Thus x is an algebraic number. Let $H(\alpha)$ be the Weil height for the number α . So $H(\alpha^n) = H^n(\alpha)$ for all $n \in \mathbb{N}$. From this we obtain that there exists a positive real number a such that $a < 1$ and for all $n \in \mathbb{N}$ we have $\alpha^n > H(\alpha^n)^a$. Now, set $\delta := x$, $q := q_n$, $\varepsilon := a\gamma$ and $u := \alpha^n$ where q_n is a suitable integer corresponding to α^n . Hence inequality (1.1) from [2] holds for infinitely many pairs (q, u) . Therefore $q_n \alpha^n x$ is a pseudo-Pisot number for infinitely many positive integers n . (A pseudo-Pisot number β is an algebraic number with $|\beta| > 1$, having all absolute values of conjugates strictly less than 1 and with $\text{Tr}_{\mathbb{Q}(\beta)/\mathbb{Q}} \in \mathbb{Z}$.) From the definition of α we have that α has a conjugate α^* with $|\alpha^*| > 1$. Thus there exists an automorphism σ of the set \mathbb{K} such that $\alpha^* = \sigma(\alpha)$ where \mathbb{K} is the Galois closure over \mathbb{Q} of the field $\mathbb{Q}(\alpha, x)$. (For more information see e.g. [10], chapter 5, page 243, lines 8–12 from the top. See also Lemma 4 from [1].) Hence for all $n \in \mathbb{N}$ the automorphism σ maps the number $q_n \alpha^n x$ to its conjugate and for infinitely many positive integers n the number $q_n \alpha^n x$ is a pseudo-Pisot number. So for infinitely many n either $q_n \alpha^n x = \sigma(q_n \alpha^n x) = q_n (\alpha^*)^n \sigma(x)$ or $1 > |\sigma(q_n \alpha^n x)| = |q_n| |\alpha^*|^n |\sigma(x)|$. But for the number α^* we have $|\alpha^*| > 1$. So the number of n such that $1 > |\sigma(q_n \alpha^n x)| = |q_n| |\alpha^*|^n |\sigma(x)|$ is finite. Therefore $x/\sigma(x) = (\alpha^*/\alpha)^n$ for infinitely many $n \in \mathbb{N}$ which is a contradiction with the fact that $|\alpha^*/\alpha|$ is a positive real number which is not equal to 1. \square

Lemma 1. *Let y be a positive real number and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers such that*

$$(3.1) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} > y + 1.$$

Then $\limsup_{n \rightarrow \infty} \left(a_n / \sum_{j=1}^{n-1} a_j \right) > y$.

Proof of Lemma 1. Let us assume that $\limsup_{n \rightarrow \infty} \left(a_n / \sum_{j=1}^{n-1} a_j \right) \leq y$. Then for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $a_n \leq \sum_{j=1}^{n-1} a_j (y + \delta)$ for every $n \geq n_0$. From

this we obtain that for all $n > n_0$

$$\begin{aligned} a_n &\leq (y + \delta) \sum_{j=1}^{n-1} a_j = (y + \delta) \left(a_{n-1} + \sum_{j=1}^{n-2} a_j \right) \leq (y + \delta) \left((y + \delta) \sum_{j=1}^{n-2} a_j + \sum_{j=1}^{n-2} a_j \right) \\ &= (y + \delta)(1 + y + \delta) \sum_{j=1}^{n-2} a_j \leq \dots \leq (y + \delta)(1 + y + \delta)^{n-n_0-1} \sum_{j=1}^{n_0} a_j. \end{aligned}$$

Hence $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \leq 1 + y$ which contradicts (3.1). \square

Proof of Theorem 3. Let N_0 be a sufficiently large positive integer. For $m \geq N_0$ set $p = p(m) = \prod_{n=1}^m [\alpha^{a_n}]$ and $N = N(m) = \sum_{n=1}^m a_n$. Then

$$(3.2) \quad \left| x - \frac{p}{\alpha^N} \right| = \left| \frac{p}{\alpha^N} \right| \cdot \left| 1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} \right|.$$

Using the inequality $|1 - t| \leq |\log t|$ for $0 < t < 1$ we deduce from the above

$$\left| 1 - \prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} \right| \leq \left| \log \left(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} \right) \right|.$$

On the other hand

$$\log \left(\prod_{n=m+1}^{\infty} \frac{[\alpha^{a_n}]}{\alpha^{a_n}} \right) = \sum_{n=m+1}^{\infty} \log \left(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}} \right),$$

where the symbol $\{\cdot\}$ stands for the fractional part. Using the inequality $|\log(1-t)| \leq |2t|$ for $0 < t < \frac{1}{2}$, and the fact that the fractional part $\{\cdot\}$ is always < 1 , we obtain

$$\sum_{n=m+1}^{\infty} \left| \log \left(1 - \frac{\{\alpha^{a_n}\}}{\alpha^{a_n}} \right) \right| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}} < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{\alpha}{\alpha - 1}.$$

From the above inequalities, (3.2) and the fact that $p/\alpha^N \leq 1$ we obtain that

$$(3.3) \quad \left| x - \frac{p}{\alpha^N} \right| < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha - 1}.$$

We shall now compare the integer $N = \sum_{n=1}^m a_n$ with a_{m+1} . From Lemma 1 we obtain that there is a $\gamma > 0$ such that $a_{m+1} \geq (1 + \gamma)N$ for infinitely many m . This and (3.3) yield that for infinitely many m with $N = \sum_{n=1}^m a_n$

$$\left| x - \frac{p}{\alpha^N} \right| < \frac{2}{\alpha^{a_{m+1}}} \cdot \frac{1}{\alpha - 1} \leq \frac{2}{\alpha^{(1+\gamma)N}} \cdot \frac{1}{\alpha - 1} \leq \frac{1}{\alpha^{(1+\gamma/2)N}}.$$

This and Theorem 2 (setting $q = 1$ in (2.1)) imply that the number x is transcendental. \square

Lemma 2. Let $\varepsilon > 0$ and $\{b_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers such that $b_n \geq n^{1+\varepsilon}$. Then $\sum_{j=n}^{\infty} 1/b_j < (1 + 2^\varepsilon/\varepsilon)/b_n^{\varepsilon/(1+\varepsilon)}$ for every $n \geq 1$.

Proof of Lemma 2. We have

$$(3.4) \quad \sum_{j=n}^{\infty} \frac{1}{b_j} = \sum_{n+j \leq b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} + \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}}.$$

We will estimate both sums on the right hand side of the equation (3.4). For the first summand we have

$$(3.5) \quad \sum_{n+j \leq b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} \leq \frac{[b_n^{1/(1+\varepsilon)}] - n + 1}{b_n} \leq \frac{b_n^{1/(1+\varepsilon)} - n + 1}{b_n}.$$

Now we will estimate the second summand.

$$(3.6) \quad \begin{aligned} \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} &\leq \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{(n+j)^{1+\varepsilon}} \\ &< \int_{[b_n^{1/(1+\varepsilon)}]}^{\infty} \frac{dx}{x^{1+\varepsilon}} = \frac{1}{\varepsilon [b_n^{1/(1+\varepsilon)}]^\varepsilon} = \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \frac{b_n^{\varepsilon/(1+\varepsilon)}}{[b_n^{1/(1+\varepsilon)}]^\varepsilon} \\ &\leq \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \left(1 + \frac{1}{[b_n^{1/(1+\varepsilon)}]}\right)^\varepsilon \leq \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \left(1 + \frac{1}{n}\right)^\varepsilon. \end{aligned}$$

From (3.4), (3.5) and (3.6) we obtain that

$$\begin{aligned} \sum_{j=n}^{\infty} \frac{1}{b_j} &= \sum_{n+j \leq b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} + \sum_{n+j > b_n^{1/(1+\varepsilon)}} \frac{1}{b_{n+j}} \\ &< \frac{b_n^{1/(1+\varepsilon)} - n + 1}{b_n} + \frac{1}{\varepsilon} \frac{1}{b_n^{\varepsilon/(1+\varepsilon)}} \left(1 + \frac{1}{n}\right)^\varepsilon \\ &= \frac{1 - n/b_n^{1/(1+\varepsilon)} + 1/b_n^{1/(1+\varepsilon)}}{b_n^{1-1/(1+\varepsilon)}} + \frac{\varepsilon^{-1} (1 + 1/n)^\varepsilon}{b_n^{\varepsilon/(1+\varepsilon)}} \\ &\leq \frac{1 + \varepsilon^{-1} (1 + 1/n)^\varepsilon}{b_n^{\varepsilon/(1+\varepsilon)}} < \frac{1 + 2^\varepsilon/\varepsilon}{b_n^{\varepsilon/(1+\varepsilon)}} \end{aligned}$$

and the proof of Lemma 2 is complete. \square

Proof of Theorem 4. Let N_0 be a sufficiently large positive integer. For $m \geq N_0$ set $p = p(m) = \prod_{n=1}^m [\alpha^{a_n}]$ and $N = N(m) = \sum_{n=1}^m a_n$. Now we proceed as in the proof of Theorem 3 to obtain that

$$\left| x - \frac{p}{\alpha^N} \right| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}}.$$

This and Lemma 2 yield that

$$\left| x - \frac{p}{\alpha^N} \right| < \sum_{n=m+1}^{\infty} \frac{2}{\alpha^{a_n}} = 2 \cdot \sum_{n=m+1}^{\infty} \frac{1}{\alpha^{a_n}} < 2 \cdot \frac{1 + 2^\varepsilon/\varepsilon}{\alpha^{a_{m+1}\varepsilon/(1+\varepsilon)}}.$$

We shall now compare the integer $N = \sum_{n=1}^m a_n$ with a_{m+1} . From Lemma 1 we obtain that there is a γ such that for infinitely many n

$$\frac{\varepsilon}{1+\varepsilon} a_{m+1} \geq \frac{\varepsilon}{1+\varepsilon} \left(1 + \frac{1}{\varepsilon} + \gamma \right) N = \left(1 + \frac{\varepsilon}{1+\varepsilon} \gamma \right) N.$$

This and Theorem 2 imply that the number x is transcendental. \square

Proof of Theorem 5. From (2.3) we obtain that there exist infinitely many n such that

$$(3.7) \quad B_n^{(2+d+\delta)^{-n}} > \max_{j=1, \dots, n-1} B_j^{(2+d+\delta)^{-j}}.$$

Otherwise there exist a positive integer n_0 such that for all $n > n_0$

$$B_n^{(2+d+\delta)^{-n}} \leq \max_{j=1, \dots, n_0-1} B_j^{(2+d+\delta)^{-j}}$$

which contradicts (2.3). The inequality (3.7) implies that for infinitely many n

$$\begin{aligned} B_n &> \left(\max_{j=1, \dots, n-1} B_j^{(2+d+\delta)^{-j}} \right)^{(2+d+\delta)^n} \\ &> \left(\max_{j=1, \dots, n-1} B_j^{(2+d+\delta)^{-j}} \right)^{(1+d+\delta)((2+d+\delta)^{n-1} + (2+d+\delta)^{n-2} + \dots + 1)} > \left(\prod_{j=1}^{n-1} B_j \right)^{1+d+\delta}. \end{aligned}$$

From this we obtain that for infinitely many n

$$(3.8) \quad B_n^{\varepsilon/(1+\varepsilon)} > \left(\prod_{j=1}^{n-1} B_j \right)^{(1+d+\delta)\varepsilon/(1+\varepsilon)}.$$

Now we proceed as in the proof of Theorem 4. Hence we obtain that for all sufficiently large m we have

$$(3.9) \quad \left| x - \frac{\prod_{k=1}^m [B_k]}{\prod_{k=1}^m B_k} \right| < \sum_{k=m+1}^{\infty} \frac{2}{B_k} = 2 \cdot \sum_{k=m+1}^{\infty} \frac{1}{B_k} < 2 \cdot \frac{1 + 2^\varepsilon/\varepsilon}{B_{m+1}^{\varepsilon/(1+\varepsilon)}} = \frac{s}{B_{m+1}^{\varepsilon/(1+\varepsilon)}}$$

where $s = 2(1 + 2^\varepsilon/\varepsilon)$. Set $\varepsilon' = \frac{1}{2}((1 + d + \delta)\varepsilon/(1 + \varepsilon) - 1 - d)$. From (2.2) we obtain that $\varepsilon' > 0$. The inequalities (2.2) and (3.8) imply that for infinitely many n

$$\frac{s}{B_n^{\varepsilon/(1+\varepsilon)}} < \frac{s}{(\prod_{j=1}^{n-1} B_j)^{(1+d+\delta)\varepsilon/(1+\varepsilon)}} = \frac{s}{(\prod_{j=1}^{n-1} B_j)^{1+d+\varepsilon'}} < \frac{1}{(\prod_{j=1}^{n-1} B_j)^{1+d+\varepsilon'/2}}.$$

From this, (3.9) and the fact that $B_k = b_k \alpha^{a_k}$ we obtain that for infinitely many n

$$\begin{aligned} \left| x - \frac{\prod_{k=1}^{n-1} [B_k]}{(\prod_{k=1}^{n-1} b_k) \alpha^{\sum_{k=1}^{n-1} a_k}} \right| &= \left| x - \frac{\prod_{k=1}^{n-1} [B_k]}{\prod_{k=1}^{n-1} b_k \alpha^{a_k}} \right| = \left| x - \frac{\prod_{k=1}^{n-1} [B_k]}{\prod_{k=1}^{n-1} B_k} \right| < \frac{s}{B_n^{\varepsilon/(1+\varepsilon)}} \\ &< \frac{1}{(\prod_{j=1}^{n-1} B_j)^{1+d+\varepsilon'/2}} = \frac{1}{(\prod_{k=1}^{n-1} b_k \alpha^{a_k})^{1+d+\varepsilon'/2}} \\ &\leq \frac{1}{\alpha^{(1+\varepsilon'/2) \sum_{k=1}^{n-1} a_k} (\prod_{k=1}^{n-1} b_k)^{1+d+\varepsilon'/2}}. \end{aligned}$$

This and Theorem 2 imply that the number x is transcendental. \square

Proof of Theorem 6. From (2.5) we obtain that there is a sufficiently small positive real number δ such that $\alpha^{(s-\delta/3)a_M} \leq b_M \leq \alpha^{(s+\delta/3)a_M}$ for all sufficiently large M . Similarly as in the proofs of Theorems 3–5 we have

$$\left| x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n} \right| < \sum_{n=m+1}^{\infty} \frac{K}{B_n}$$

for all sufficiently large positive integers m where K is a suitable positive real constant which does not depend on m . From this and the fact that $\alpha^{(s-\delta/3)a_M} \leq b_M$ we obtain that for all sufficiently large positive integers m

$$\begin{aligned} (3.10) \quad \left| x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n} \right| &< \sum_{n=m+1}^{\infty} \frac{K}{B_n} \leq \sum_{n=m+1}^{\infty} \frac{1}{\alpha^{(s+1-\delta/2)a_n}} \\ &\leq \frac{1}{\alpha^{(s+1-\delta/2)a_{m+1}}} \cdot \frac{1}{1 - 1/\alpha^{s+1-\delta/2}} \\ &= \frac{1}{\alpha^{(s+1-\delta/2)a_{m+1}}} \cdot \frac{\alpha^{s+1-\delta/2}}{\alpha^{s+1-\delta/2} - 1} \leq \frac{1}{\alpha^{(s+1-\delta)a_{m+1}}}. \end{aligned}$$

From (2.4) and Lemma 1 we obtain that for infinitely many m

$$(3.11) \quad a_{m+1} > \left(1 + \frac{sd}{s+1} + \delta' \right) \sum_{n=1}^m a_n$$

where δ' is a real number such that $0 < \delta' < \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} - 2 - sd/(s+1)$. From (3.11) and the fact that δ is a sufficiently small positive real number we obtain that for infinitely many m

$$\begin{aligned} (s+1-\delta)a_{m+1} &> (s+1-\delta)\left(1 + \frac{sd}{s+1} + \delta'\right) \sum_{n=1}^m a_n \\ &= \left(s+1+sd + (s+1)\delta' - \delta\left(1 + \frac{sd}{s+1} + \delta'\right)\right) \sum_{n=1}^m a_n \\ &= \left(1 + (s+\delta)(d+1) + \delta' + s\delta' - \delta\left(d+2 + \frac{sd}{s+1} + \delta'\right)\right) \sum_{n=1}^m a_n \\ &\geq (1 + (s+\delta)(d+1) + \delta') \sum_{n=1}^m a_n. \end{aligned}$$

From this, (3.10) and the fact that $b_M \leq \alpha^{(s+\delta/3)a_M}$ we obtain that for infinitely many m

$$\begin{aligned} \left|x - \frac{\prod_{n=1}^m [B_n]}{(\prod_{n=1}^m b_n) \alpha^{\sum_{n=1}^m a_n}}\right| &= \left|x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n}\right| \leq \frac{1}{\alpha^{(s+1-\delta)a_{m+1}}} \\ &\leq \frac{1}{\alpha^{((s+\delta)(1+d)+1+\delta') \sum_{n=1}^m a_n}} \\ &= \frac{1}{\alpha^{((s+\delta/2)(1+d+\delta'/(s+\delta+1))+\frac{1}{2}\delta(1+d+\delta'/(s+\delta+1))+1+\delta'/(s+\delta+1)) \sum_{n=1}^m a_n}} \\ &\leq \frac{1}{(\prod_{n=1}^m b_n)^{1+d+\delta'/(s+\delta+1)} \alpha^{(1+\delta'/(s+\delta+1)) \sum_{n=1}^m a_n}}. \end{aligned}$$

This and Theorem 2 imply that the number x is transcendental. \square

Proof of Theorem 7. From (2.8) we obtain that there is a sufficiently small positive real number δ such that $\alpha^{(s-\delta/3)a_M} \leq b_M \leq \alpha^{(s+\delta/3)a_M}$ for all sufficiently large M . Similarly as in the proofs of Theorems 3–6 we have

$$\left|x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n}\right| < \sum_{n=m+1}^{\infty} \frac{K}{B_n}$$

where K is a suitable positive real constant which does not depend on m . From this, Lemma 2, (2.7) and the fact that $\alpha^{(s-\delta/3)a_M} \leq b_M$ we obtain that for all sufficiently large positive integers m

$$\begin{aligned} (3.12) \quad \left|x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n}\right| &< \sum_{n=m+1}^{\infty} \frac{K}{B_n} \leq \sum_{n=m+1}^{\infty} \frac{1}{\alpha^{(s+1-\delta/2)a_n}} \\ &\leq \frac{1 + 2^\varepsilon/\varepsilon}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta/2)a_{m+1}}} \leq \frac{1}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta)a_{m+1}}}. \end{aligned}$$

From (2.6) and Lemma 1 we obtain that for infinitely many m

$$(3.13) \quad a_{m+1} > \left(\left(1 + \frac{sd}{s+1} \right) \frac{1+\varepsilon}{\varepsilon} + \delta' \right) \sum_{n=1}^m a_n$$

where δ' is a real number such that $0 < \delta' < \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} - 1 - (1 + (sd/(s+1)) \times (1+\varepsilon)/\varepsilon)$. From (3.13) and the fact that δ is a sufficiently small positive real number we obtain that for infinitely many m

$$\begin{aligned} \frac{\varepsilon}{1+\varepsilon} (s+1-\delta) a_{m+1} &> \frac{\varepsilon}{1+\varepsilon} (s+1-\delta) \left(\left(1 + \frac{sd}{s+1} \right) \frac{1+\varepsilon}{\varepsilon} + \delta' \right) \sum_{n=1}^m a_n \\ &= \left((s+\delta)(d+1) + 1 + \frac{\varepsilon(s+1)}{2(1+\varepsilon)} \delta' \right) \sum_{n=1}^m a_n. \end{aligned}$$

From this, (3.12), and the inequality $b_M \leq \alpha^{(s+\delta/3)a_M}$ we obtain that for infinitely many m

$$\begin{aligned} \left| x - \frac{\prod_{n=1}^m [B_n]}{\left(\prod_{n=1}^m b_n \right) \alpha^{\sum_{n=1}^m a_n}} \right| &= \left| x - \frac{\prod_{n=1}^m [B_n]}{\prod_{n=1}^m B_n} \right| \leq \frac{1}{\alpha^{(\varepsilon/(1+\varepsilon))(s+1-\delta)a_{m+1}}} \\ &\leq \frac{1}{\alpha^{((s+\delta)(1+d)+1+\frac{1}{2}\varepsilon\delta'(s+1)/(1+\varepsilon)) \sum_{n=1}^m a_n}} \\ &= \frac{1}{\alpha^{((s+\delta/2)(1+d+\varepsilon')+1+\varepsilon'+(\frac{1}{2}\varepsilon\delta'(s+1)/(1+\varepsilon)+\frac{1}{2}\delta(1+d)-\varepsilon'(1+s+\delta/2))) \sum_{n=1}^m a_n}} \\ &\leq \frac{1}{\left(\prod_{n=1}^m b_n \right)^{(1+d+\varepsilon')} \alpha^{(1+\varepsilon') \sum_{n=1}^m a_n}} \end{aligned}$$

where ε' is a real number such that $\frac{1}{2}\delta'\varepsilon(s+1)/(1+\varepsilon) + \frac{1}{2}\delta(1+d) > \varepsilon'(1+s+\frac{1}{2}\delta) > 0$. This and Theorem 2 imply that the number x is transcendental. \square

Acknowledgement. The authors would like to thank James E. Carter of the College of Charleston for helping us with our English. The authors also thank Pietro Corvaja of the University of Udine and the referee for their valuable suggestions.

References

- [1] *P. Corvaja, J. Hančl*: A transcendence criterion for infinite products. *Atti Acad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.* **18** (2007), 295–303. [zbl](#) [MR](#)
- [2] *P. Corvaja, U. Zannier*: On the rational approximations to the powers of an algebraic number: solution of two problems of Mahler and Mendès France. *Acta Math.* **193** (2004), 175–191. [zbl](#) [MR](#)
- [3] *P. Corvaja, U. Zannier*: Some new applications of the subspace theorem. *Comp. Math.* **131** (2002), 319–340. [zbl](#) [MR](#)

- [4] *P. Erdős*: Some problems and results on the irrationality of the sum of infinite series. *J. Math. Sci.* *10* (1975), 1–7. zbl MR
- [5] *M. Genčev*: Evaluation of infinite series involving special products and their algebraic characterization. *Math. Slovaca* *59* (2009), 365–378. zbl MR
- [6] *J. Hančl, R. Nair, J. Šustek*: On the Lebesgue measure of the expressible set of certain sequences. *Indag. Math., New Ser.* *17* (2006), 567–581. zbl MR
- [7] *J. Hančl, P. Rucki, J. Šustek*: A generalization of Sándor’s theorem using iterated logarithms. *Kumamoto J. Math.* *19* (2006), 25–36. zbl MR
- [8] *J. Hančl, J. Štěpnička, J. Šustek*: Linearly unrelated sequences and problem of Erdős. *Ramanujan J.* *17* (2008), 331–342. zbl MR
- [9] *D. Kim, J. K. Koo*: On the infinite products derived from theta series I. *J. Korean Math. Soc.* *44* (2007), 55–107. zbl MR
- [10] *S. Lang*: Algebra (3rd ed.). Graduate Texts in Mathematics. Springer, New York, 2002. zbl MR
- [11] *M. A. Nyblom*: On the construction of a family of transcendental valued infinite products. *Fibonacci Q.* *42* (2004), 353–358. zbl MR
- [12] *Y. Tachiya*: Transcendence of the values of infinite products in several variables. *Result. Math.* *48* (2005), 344–370. zbl MR
- [13] *P. Zhou*: On the irrationality of a certain multivariable infinite product. *Quaest. Math.* *29* (2006), 351–365. zbl MR
- [14] *Y. Ch. Zhu*: Transcendence of certain infinite products. *Acta Math. Sin.* *43* (2000), 605–610. (In Chinese. English summary.) zbl MR

Authors’ addresses: Jaroslav Hančl, Department of Mathematics and Centre of Excellence IT4Innovation, division of UO, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail: hanc1@osu.cz; Ondřej Kolouch, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail: ondrej.kolouch@osu.cz; Simona Pulcerová, Department of Mathematical Methods in Economics, Faculty of Economics, VŠB-Technical University of Ostrava, Sokolská třída 33, 701 21 Ostrava 1, Czech Republic, e-mail: simona.pulcerova@vsb.cz; Jan Štěpnička, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail: jan.stepnicka@osu.cz.